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THE VALUES OF NONSTANDARD EXCHANGE ECONOMIES†

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ABSTRACT
R. J. Aumann has stated and rigorously proved the value equivalence theorem for exchange economies with a non-atomic continuum of traders. The analogous result is established here for nonstandard exchange economies using Abraham Robinson's calculus of infinitesimals. The proof is patterned after a short heuristic argument given by Aumann.

I. Introduction

In [1], Aumann stated and rigorously proved the value equivalence theorem for exchange economies with a non-atomic continuum of traders. Since the proof was complex and involved, he also gave a heuristic argument. His nonrigorous argument is based on a naive notion of infinitesimals. We shall prove a value equivalence theorem for nonstandard exchange economies, using Robinson's calculus of infinitesimals, i.e., nonstandard analysis. Our argument follows closely Aumann's original intuitive argument.

Nonstandard exchange economies and the associated nonstandard concepts of the core and competitive equilibrium were first defined in Brown–Robinson [3]. In that paper, they proved the equivalence between the nonstandard core and the set of nonstandard competitive allocations, i.e., an allocation is in the core iff it is a competitive allocation. In a second paper [4], using the core equivalence theorem for nonstandard exchange economies, they showed that core allocations in large standard economies are approximate competitive allocations. The interested reader is referred to either of these two papers for an introduction to nonstandard analysis and a discussion of nonstandard exchange economies.

We should note that the existence of a value allocation follows from the value equivalence theorem, and the existence of nonstandard competitive equilibria shown by Brown in [5].

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Before stating and proving our major theorem, we shall need the following definitions.

II. Definitions

Let $^*R_d$ be the d-fold Cartesian product of $^*R$, the nonstandard extension of R, and $^*\Omega_d$ be the positive orthant of $^*R_d$. If $\bar{x}$ and $\bar{y}$ are vectors in $^*R_d$, then we shall write $\bar{x} \simeq \bar{y}$ when the distance between x and y is infinitesimal in the metric defined by the sup norm. $\bar{x} \geqq \bar{y}$ means $x_i \geqq y_i$ for all i; $\bar{x} > \bar{y}$ means $\bar{x} \geqq \bar{y}$ and $x_i > y_i$ for some i; $\bar{x} \gg \bar{y}$ means $x_i > y_i$ for all i. $\bar{x} \gtrsim \bar{y}$ means $x_i \geqq y_i$ or $x_i \simeq y_i$ for all i; $\bar{x} \gtrapprox \bar{y}$ means $\bar{x} \gtrsim \bar{y}$ and x_i is greater than y_i by a noninfinitesimal amount for some i; $\bar{x} \ggg \bar{y}$ means x_i is greater than y_i by a noninfinitesimal amount for all i. The vector which is 0 in all components except the i-th and 1 in the i-th is denoted by $\bar{e}^i$.

By the norm of a vector $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_d)$, we mean the sup-norm $\| x \| = \max_{1 \leq j \leq d} | \bar{x}_j |$. A nonstandard vector is said to be finite or near standard if its sup-norm is less than some standard number. If $\bar{x}$ is a finite vector, then there exists a unique standard vector, called the standard part of $\bar{x}$, denoted by $°\bar{x}$, where $°\bar{x} \simeq \bar{x}$.

$^*N$ will denote the nonstandard extension of N, the integers, and $^*N - N$ is the set of infinite integers. If δ is an infinitesimal, then we shall often write $\delta \simeq 0$, with similar notation used for infinitesimal vectors.

Let T be an internal star-finite set. T will be called the set of traders or agents. We shall always assume that T is infinite, i.e., the internal cardinality $| T | = \omega \in {}^*N - N$.

A *coalition* is an internal subset of traders.

A coalition S is *negligible* if $| S |/\omega \simeq 0$.

All agents are assumed to have the same *consumption set* which is $^*\Omega_d$.

An *assignment* is an internal map from T into $^*\Omega_d$.

A trader is defined by his *initial endowment*, an element of $^*\Omega_d$, and his *preference relation*, a binary relation on $^*\Omega_d$.

A *nonstandard exchange economy*, $\mathscr{E}$, is a pair $\langle I, P \rangle$ where $I(t)$ is an assignment and $P(t)$ is an internal map from T into the family of internal binary relations on $^*\Omega_d$. We will often denote $P(t)$ as $>_t$. That is, $>_t$ is the preference relation of trader t and $I(t)$ is his initial endowment.

An *allocation*, X, is an assignment such that $X(t)$ is finite for each $t \in T$. It follows that the norm $\| X(t) \|$ is uniformly bounded.

An assignment X is *feasible for a coalition S* if

$$\frac{1}{|S|}\sum_{t\in S} X(t) \simeq \frac{1}{|S|}\sum_{t\in S} I(t).$$

An assignment X is *strictly feasible for a coalition S* if

$$\frac{1}{|S|}\sum_{t\in S} X(t) = \frac{1}{|S|}\sum_{t\in S} I(t), \quad \text{i.e.,} \quad \sum_{t\in S} X(t) = \sum_{t\in S} I(t).$$

If $S = T$, then we will say that X is feasible or strictly feasible.

If $\bar{x}, \bar{y} \in {}^*\Omega_d$, then $\bar{x} >_t > \bar{y}$ iff for all $\bar{w} \simeq \bar{x}$ and for all $\bar{z} \simeq \bar{y}$, $\bar{w} >_t \bar{z}$. If $\bar{p} \in {}^*\Omega_d$, then

$$B_{\bar{p}}(t) = \{\bar{x} \in {}^*\Omega_d : \bar{p}\cdot\bar{x} \lesssim p \cdot I(t)\}.$$

Given a feasible allocation $X(t)$, $\langle X, p\rangle$ is a *competitive equilibrium* for the economy $\mathscr{E} = \langle I(t), \{>_t\}_{t\in T}\rangle$ iff $\bar{p}$ is finite, $\bar{p} \gtrsim 0$ and there exists a coalition T_0, with $|T_0|/|T| \simeq 0$, such that for all $t \in T - T_0$, $X(t) \in B_{\bar{p}}(t)$ and $y \notin B_{\bar{p}}(t)$ for any $\bar{y} >_t > X(t)$. If $\langle X, \bar{p}\rangle$ is a competitive equilibrium, then we say that X is a *competitive allocation* and $\bar{p}$ is a *competitive price system.*

If $>_t$ is a preference relation over ${}^*\Omega_d$, then a utility (function) for $>_t$ is an internal map $u_t : {}^*\Omega_d \to {}^*R$ such that for all $\bar{x}, \bar{y} \in {}^*\Omega_d$, $\bar{x} >_t \bar{y}$ iff $u_t(\bar{x}) > u_t(\bar{y})$. If $\{>_t\}_{t\in T}$ is an internal family of preference relations, then an internal family of utilities $\{u_t\}_{t\in T}$ is said to represent $\{>_t\}_{t\in T}$ if for each $t \in T$, u_t is a utility function for $>_t$.

If $\{u_t\}_{t\in T}$ is a representing family for $\{>_t\}_{t\in T}$, then $\mathscr{E}$ can be expressed as $\langle I(t), \{u_t\}_{t\in T}\rangle$. We now use $\mathscr{E}$ to define a star-finite internal game $\mathscr{G}$, in characteristic form, over T. Let $\mathscr{I}(T)$ be the family of internal subsets of T or coalitions of T. Then $\mathscr{G} = \langle V, \mathscr{I}(T)\rangle$ where $V : \mathscr{I}(T) \to {}^*R$ is defined as follows: Given $S \in \mathscr{I}(T)$, $V(S) = \max (1/\omega)\Sigma_{t\in S} u_t(Y(t))$ where Y ranges over all assignments with $\Sigma_{t\in S} Y(t) \leqq \Sigma_{t\in S} I(t)$.

The Shapley-value is an *a priori* evaluation of a finite game for each player. A brief discussion of the Shapley-value and its properties is given in Appendix A of [2]. By transfer, the Shapley-value of $\mathscr{G}$ is well defined and we denote it by φ_V. Of course, $\mathscr{G}$ and hence φ_V depends on $\{u_t\}_{t\in T}$ and in general different representing families define different games.

A value allocation with respect to $\mathscr{E} = \langle I(t), \{u_t\}_{t\in T}\rangle$ is a feasible allocation X such that for each $S \in \mathscr{I}(T)$, $\Sigma_{t\in S}\varphi_V(t) \simeq (1/\omega)\Sigma_{t\in S} u_t(X(t))$. It is important to note that a value allocation depends on the representation of the preferences $\{>_t\}_{t\in T}$.

Debreu [6] defines a (standard) *smooth preference relation* over Ω_d as a binary relation over Ω_d having a quasi-concave C^2 standard utility with a strictly positive gradient and indifference surfaces with everywhere positive Gaussian curvature.

A uniformly smooth family of (standard) preferences was first defined by Aumann [1]. In particular, he considered the following example of a uniformly smooth family, which we shall call a precompact smooth family. A family of standard utilities $\{u_s\}_{s\in S}$ on Ω_d (i.e., for all $s \in S$, $u_s : \Omega_d \to R$) is said to be a *precompact smooth family* if:

a) Each u_s is smooth (in the sense of Debreu), the bounds on compact sets being uniform over S.

b) The u_s are bounded over Ω_d uniformly in s.

c) $\{u_s\}_{s\in S}$ is contained in a compact subset of the space $\mathcal{U}$ of C^2 utility functions on Ω_d endowed with the topology of C^2-uniform uniform convergence on compact sets.

If $\bar{x} \in \Omega_d$, then a C^1 utility (function) u on Ω_d is called concave over Ω_d at $\bar{x}$ (or *concave at* $\bar{x}$) if for all $\bar{y} \in \Omega_d$, $u(\bar{y}) - u(\bar{x}) \leqq \nabla u(\bar{x}) \cdot (\bar{y} - \bar{x})$ where $\nabla u(\bar{x})$ is the gradient of u evaluated at $\bar{x}$.

III. Assumptions

1) $|T| = \omega \in {}^*N - N$.

2) $I(t)$ is an allocation.

3) $(1/\omega)\Sigma_{t\in T} I(t) \gg \bar{0}$.

4) For all $t \in T$, $I(t) \neq \bar{0}$.

5) The preferences $\{>_t\}_{t\in T}$ are represented by an internal family of utility functions $\{u_t\}_{t\in T}$ where $\{u_t\}_{t\in T}$ is contained in the nonstandard extension of a family $\mathcal{F}$ of standard utility functions.

Aumann in [1], lemma 15.1, has shown that if $\{u_s\}_{s\in S}$ is a precompact smooth family of standard utility functions, then for every $\gamma > 0$ there exists a family of utilities $\{\tilde{u}_s\}_{s\in S}$ such that:

a) The $\tilde{u}_s$ are uniformly bounded.

b) The $\tilde{u}_s$ are C^1 on Ω_d.

c) The gradients $\nabla \tilde{u}_s$ are, on compact sets, uniformly bounded and uniformly positive, i.e., for every compact set K there exists vectors $\bar{a}$ and $\bar{b}$, where $\bar{a} \gg \bar{0}$ and $\bar{b} \gg \bar{0}$ such that for all $s \in S$, for all $\bar{y} \in K$, $\bar{a} \leqq \nabla \tilde{u}_s(\bar{y}) \leqq \bar{b}$.

d) Each $\tilde{u}_s$ is concave over Ω_d at each $\bar{x}$ such that $\|\bar{x}\| \leqq \gamma$.

A family of utilities satisfying (a), (b), (c), and (d) above shall be called *essentially concave* in $B_\gamma(\bar{0}) = \{\bar{x} \in \Omega_d : \|\bar{x}\| \leqq \gamma\}$.

By a representing family of utilities $\{u_t\}_{t\in T}$ we shall mean a family satisfying Assumption 5 such that the standard utilities $\bar{u}_s$ in $\mathscr{F}$ satisfy at least Properties a, b, and c above and $\mathscr{F}$ is compact in the topology of C^1-uniform convergence on compact sets. If Property d also holds, we shall say that the family $\{u_t\}_{t\in T}$ is essentially concave in $^*B_\gamma(\bar{0})$.

IV. Statement of the Theorem

Let $\mathscr{E} = \langle I(t), \{>_t\}_{t\in T}\rangle$ be a nonstandard exchange economy satisfying the assumptions in Section III.

THEOREM. *a) Given a nonstandard competitive allocation X for $\mathscr{E}$, there is a $\gamma > 0$ such that if $\{u_t\}_{t\in T}$ is a representing family of utilities essentially concave in $^*B_\gamma(\bar{0})$, then there is an internal family of weights $\{\alpha_t\}_{t\in T}$ with $\alpha_t \gtrapprox 0$ for each t such that X is a value allocation with respect to the family of utilities $\{\alpha_t u_t\}_{t\in T}$.*

b) If $\{u_t\}_{t\in T}$ is a representing family of utilities and X is a value allocation with respect to $\{u_t\}_{t\in T}$, then X is a competitive allocation.

V. Proof of Theorem

LEMMA 1. *Let $\{u_t\}_{t\in T}$ be a representing family of utilities. Given a coalition $S \subset T$ and an allocation $Z(t)$, let $\mathscr{C} = \{Y : Y$ is an assignment and $\Sigma_{t\in S} Y(t) \leqq \Sigma_{t\in S} Z(t)\}$. If $X \in \mathscr{C}$ and $\Sigma_{t\in S} u_t(Y(t)) \leqq \Sigma_{t\in S} u_t(X(t))$ for all $Y \in \mathscr{C}$, then*

i) *For every $\nu \in {}^*N - N$, $\max_{t\in S} \|X(t)\| < \nu$. Thus $\max_{t\in S}\|X(t)\|$ is finite.*

ii) *There exists $\bar{p} \in {}^*\Omega_d$ such that*

α) $\bar{p}$ is finite and $\bar{p} \gtrapprox \bar{0}$,

β) Given $t \in S$, if $X_j(t) \neq \bar{0}$, then $\bar{p}_j = \nabla_j u_t(X(t))$, and if $X_j(t) = 0$, then $\bar{p}_j \geqq \nabla_j u_t(X(t))$.

PROOF. Suppose for some $s \in S$ and $j \leqq d$, $X_j(s) \geqq \nu$ where $\nu \in {}^*N - N$. Then $|S| \in {}^*N - N$ since Z is an allocation and $X \in \mathscr{C}$. Let $A_n = \{t \in S : \|X(t)\| < n\}$. If $|A_n|$ is finite for all $n \in N$, then there exists a $\rho \in {}^*N - N$ such that $|A_\rho|/|S|^{1/2} \simeq 0$ and $\rho/|S|^{1/2} \simeq 0$. Let $\|\bar{x}\|_1 = \Sigma_{j=1}^d \bar{x}_j$ if $\bar{x} \in {}^*\Omega_d$. Since preferences are monotonic, we have $\Sigma_{t\in S} X(t) = \Sigma_{t\in S} Z(t)$ so $\Sigma_{t\in S}\|X(t)\|_1 = \Sigma_{t\in S}\|Z(t)\|_1$. But

$$\frac{1}{|S|}\sum_{t\in S}\|X(t)\|_1 = \frac{1}{|S|}\sum_{t\in A_\rho}\|X(t)\|_1$$

$$+\frac{1}{|S|}\sum_{t\in S-A_\rho}\|X(t)\|_1 \simeq \frac{1}{|S|}\sum_{t\in S-A_\rho}\|X(t)\|_1 \geqq \frac{|S-A_\rho|}{|S|}\rho \geqq \frac{1}{2}\rho.$$

This contradicts the finiteness of $(1/|S|)\Sigma_{t\in S}\|Z(t)\|_1$. Hence, for some $n_0 \in N$, $|A_{n_0}| \in {}^*N - N$. Let $\theta = \min(\nu, |A_{n_0}|)$; choose $B \subset A_{n_0}$ with $|B| = \theta$, and define a new allocation

$$W(t) = \begin{cases} X(t) + \bar{e}_j & \text{for } t \in B \\ X(t) - \theta\bar{e}_j & \text{for } t = s \\ X(t) & \text{otherwise.} \end{cases}$$

Since the u_t are standardly bounded, $u_s(X(s))$ and $u_s(W(s))$ differ by only a finite amount. The u_t are near standard on any compact standard set, hence they are near standard on $(\bar{x} \in {}^*\Omega_d : \|\bar{x}\| \leqq \max_{t\in B}\|W(t)\|)$. Therefore, $\min_{t\in B}[u_t(W(t)) - u_t(X(t))] \gtrsim 0$, and $\Sigma_{t\in S}u_t(W(t)) \ggg \Sigma_{t\in S}u_t(X(t))$ which contradicts the definition of $X(t)$. Thus, for every $\nu \in {}^*N - N$, $\max_{t\in S}\|X(t)\| < \nu$.

For part (ii) of the lemma, suppose for some $j \leqq d$ and $s \in S$ that $X_j(s) > 0$. Define a vector $\Delta\bar{x}$ where $\Delta\bar{x}_i = 0$ for $i \neq j$ and $\Delta\bar{x}_j = \delta$ with $0 < \delta < X_j(s)$. If $r \in S$ and $r \neq s$, then the assignment $Y(t)$ where $Y(t) = X(t)$ for $t \neq s$ or r; $Y(s) = X(s) - \Delta\bar{x}$; and $Y(r) = X(r) + \Delta\bar{x}$ belongs to $\mathscr{C}$. Therefore, $\Sigma_{t\in S}u_t(Y(t)) \leqq \Sigma_{t\in S}u_t(X(t))$. Thus,

$$\frac{u_r(X(r)+\Delta\bar{x}) - u_r(X(r))}{\delta} \leqq \frac{u_s(X(s)) - u_s(X(s)-\Delta\bar{x})}{\delta},$$

which implies that $\nabla_j u_s(X(s)) \geqq \nabla_j u_r(X(r))$. If $X_j(t) = 0$ for all $t \in S$, let $\bar{p}_j = \max_{t\in S}\nabla_j u_t(X(t))$. If there is a $t \in S$ with $X_j(t) > 0$, let $\bar{p}_j = \nabla_j u_t(X(t))$. Clearly, $\bar{p}$ is well defined and $\bar{p}_j \geqq \nabla_j u_t(X(t)) \geqq 0$ for all $j \leqq d$ and for all $t \in S$.

By part (i), $X(t)$ is finite for all $t \in S$, thus, for all $j \leqq d$ and $s \in S$, $\nabla_j u_s(X(s)) \simeq \nabla_j^0 u_s({}^0X(s)) > 0$. Hence, for each $j \leqq d$, $\bar{p}_j \gtrsim 0$ and $\max_{s\in S}\nabla_j u_s(X(s))$ is finite, i.e., $\bar{p}$ is finite.

Definition. Given $S, Z, X, \{u_t\}_{t\in T}$, and $\bar{p}$ as in Lemma 1, we write $(X, \bar{p}) \in \mathscr{M}(s, Z, \{u_t\}_{t\in T})$, and we say that X maximizes utility for S.

Proposition 1. *Let $\{u_t\}_{t\in T}$ be a representing family of utilities, i.e., $\mathscr{F}$ satisfies the assumptions of Section III. Given a coalition $S \subset T$ with $|S| \in {}^*N - N$ and an allocation $Z(t)$ with*

$$\frac{1}{|S|}\sum_{t\in S} Z(t) \gtrsim 0,$$

let $(X, \bar{p}) \in \mathscr{M}(S, Z, \{u_t\}_{t\in T})$. Then for each $\bar{y} \in {}^*\Omega_d$ and $t \in S$,

$$u_t(\bar{y}) - u_t(X(t)) \lesssim \bar{p}\cdot(\bar{y} - X(t)).$$

For the proof we need the following lemma given the hypotheses of the Proposition.

LEMMA 2. *If $\bar{A}$ is a finite vector, there is an allocation W such that $\Sigma_{t\in S} W(t) = \bar{A}$, for each $t \in S$ and $j \leqq d$, $W_j(t) \simeq 0$ and $W_j(t) = 0$ if $X_j(t) \simeq 0$, and moreover*

$$\sum_{t\in S} u_t(X(t) + W(t)) - u_t(X(t)) \simeq \bar{p} \cdot \bar{A}.$$

PROOF. Since the utilities are increasing

$$\bar{z} \equiv \frac{1}{|S|}\sum_{t\in S} X(t) = \frac{1}{|S|}\sum_{t\in S} Z(t) \gg 0.$$

For each $j \leqq d$, let $S_j = \{s \in S : X_j(s) \geqq \bar{z}_j/2\}$. Then $|S_j| \in {}^*N - N$. Let $W_j(t) = \bar{A}_j/|S_j|$ if $t \in S_j$, and let $W_j(t) = 0$ otherwise. By Lemma 1 and the properties of $\mathcal{F}$, there exist for each $t \in S$ a vector $\bar{\varepsilon}(t) \simeq 0$ such that

$$u_t(X(t) + W(t)) - u_t(X(t)) = (\bar{p} + \bar{\varepsilon}(t)) \cdot W(t).$$

Letting $\varepsilon_0 = \max_{t\in S} \|\bar{\varepsilon}(t)\|$, we have

$$\left|\sum_{t\in S} \bar{\varepsilon}(t) \cdot W(t)\right| \leqq \varepsilon_0 \sum_{t\in S}\sum_{j=1}^{d} |W_j(t)| = \varepsilon_0 \sum_{j=1}^{d} |\bar{A}_j| \simeq 0.$$

The rest is clear.

PROOF OF PROPOSITION. Assume there is a $\bar{y} \in {}^*\Omega_d$ and a $t_0 \in S$ for which the proposition is false. The utilities are bounded, $X(t_0)$ is finite, and $\bar{p} \gg 0$, so $\bar{y}$ is finite. Choose $W(t)$ as in Lemma 2 for $t \in S - \{t_0\}$ so that $\Sigma_{t\in S-\{t_0\}} W(t) = \bar{y} - X(t_0)$. Then

$$\sum_{t\in S-\{t_0\}} u_t(X(t) - W(t)) - u_t(X(t)) \simeq -\bar{p} \cdot (\bar{y} - X(t_0)),$$

while by assumption,

$$u_{t_0}(X(t_0) + (\bar{y} - X(t_0)) - u_{t_0}(X(t_0)) \gtrsim \bar{p} \cdot (\bar{y} - X(t_0)).$$

This contradicts the definition of X.

LEMMA 3. *Given $S \subset T$ where $|S| \in {}^*N - N$, let $(Y, \bar{p}) \in \mathcal{M}(S, I, \{u_t\}_{t\in T})$. Assume $(1/|S|)\Sigma_{t\in S} Y(t) \gg 0$. If X is a feasible allocation for S such that $(1/|S|)\Sigma_{t\in S} u_t(X(t)) \simeq (1/|S|)\Sigma_{t\in S} u_t(Y(t))$, then there exists an $S_0 \subset S$ with $|S_0|/|S| \simeq 0$ so that for all $t \in S - S_0$, $u_t(X(t)) \simeq u_t(Y(t)) + \bar{p} \cdot (X(t) - Y(t))$.*

PROOF. Since X and Y are feasible on S, $\bar{p}\cdot(1/|S|)\Sigma_{t\in S}(X(t)-Y(t))\simeq 0$. Let

$$S_n=\{t\in S: u_t(X(t))-u_t(Y(t))\leqq \bar{p}\cdot(X(t)-Y(t))-1/n\}.$$

Clearly, $S_n\subset S_{n+1}$ for each $n\in {}^*N$. If $n\in N$ and $|S_n|/|S|\geqq 1/n$, then

$$\frac{1}{|S|}\sum_{t\in S}(u_t(X(t))-u_t(Y(t)))\lesssim\frac{1}{|S|}\sum_{t\in S_n}(\bar{p}\cdot(X(t)-Y(t))-1/n)$$

$$+\frac{1}{|S|}\sum_{t\in S-S_n}\bar{p}\cdot(X(t)-Y(t))=\bar{p}\cdot\left(\frac{1}{|S|}\sum_{t\in S}(X(t)-Y(t))\right)-\frac{|S_n|}{|S|}\cdot\frac{1}{n}\lesssim-\frac{1}{n^2}.$$

Hence,

$$\frac{1}{|S|}\sum_{t\in S}u_t(X(t))\lesssim\frac{1}{|S|}\sum_{t\in S}u_t(Y(t))-1/n^2.$$

But this is impossible. Therefore, there is a $\nu\in {}^*N-N$ with $|S_\nu|/|S|<1/\nu$. It follows from Proposition 1 that for all $t\in S-S_\nu$, $u_t(X(t))-u_t(Y(t))\simeq \bar{p}\cdot(X(t)-Y(t))$.

LEMMA 4. *Let S be a subset of T with $|S|\in {}^*N-N$, and let X be an allocation which is feasible on S such that $(1/|S|)\Sigma_{t\in S}X(t)\gg\bar{0}$. Then there is an allocation $Z(t)$ which is strictly feasible on S such that $Z(t)\simeq X(t)$ for all $t\in S$.*

PROOF. Let $a_j=(1/|S|)\Sigma_{t\in S}X_j(t)$ and let $A_j=\{t\in S: X_j(t)\geqq a_j/2\}$. Then $|A_j|/|S|\not\simeq 0$. For each $t\in S$ and $j\leqq d$, let

$$Z_j(t)=\begin{cases}X_j(t)+\dfrac{1}{|A_j|}\displaystyle\sum_{s\in S}(I_j(s)-X_j(s)) & \text{if } t\in A_j\\[2ex] X_j(t) & \text{otherwise.}\end{cases}$$

Since $|S|/|A_j|$ is finite and $(1/|S|)\Sigma_{t\in S}(I_j(t)-X_j(t))\simeq 0$, $Z_j(t)\simeq X_j(t)$ for all $j\leqq d$ and all $t\in S$. Clearly, $Z(t)$ is strictly feasible on S.

LEMMA 5. *Let $(X,\bar{p})\in\mathcal{M}(T,I,\{u_t\}_{t\in T})$ and let S be a subset of T such that $|S|\in {}^*N-N$, X is feasible on S, and $(1/|S|)\Sigma_{t\in S}X(t)\gg\bar{0}$. Let $(Y,\bar{q})\in\mathcal{M}(S,I,\{u_t\}_{t\in T})$. Then $\bar{p}\simeq\bar{q}$.*

PROOF. By Lemma 4, there is an allocation $Z(t)$ which is strictly feasible on S such that $Z(t)\simeq X(t)$, for all $t\in S$. Therefore,

$$\frac{1}{|S|}\sum_{t\in S}u_t(X(t))\simeq\frac{1}{|S|}\sum_{t\in S}u_t(Z(t))\leqq\frac{1}{|S|}\sum_{t\in S}u_t(Y(t)).$$

On the other hand, $(X, \bar{p}) \in \mathcal{M}(S, X, \{u_t\}_{t \in T})$ and $(1/|S|)\Sigma_{t \in S} Y(t) = (1/|S|)\Sigma_{t \in S} I(t) \simeq (1/|S|)\Sigma_{t \in S} X(t)$. Again by Lemma 4 (using X instead of I) there is an allocation W with $\Sigma_{t \in S} W(t) = \Sigma_{t \in S} X(t)$ and $W(t) \simeq Y(t)$ for each $t \in S$. Therefore,

$$\frac{1}{|S|}\sum_{t \in S} u_t(Y(t)) \simeq \frac{1}{|S|}\sum_{t \in S} u_t(W(t)) \leqq \frac{1}{|S|}\sum_{t \in S} u_t(X(t)).$$

Thus, $(1/|S|)\Sigma_{t \in S} u_t(X(t)) \simeq (1/|S|)\Sigma_{t \in S} u_t(Y(t))$, so by Lemma 3 there exists $S' \subset S$ with $|S'|/|S| \simeq 0$, such that for all $t \in S - S'$, $u_t(X(t)) - u_t(Y(t)) \simeq \bar{q} \cdot (X(t) - Y(t))$. Applying the same lemma to $Y(t)$ and $(X, \bar{p})$ we see that for some $S'' \subset S$ with $|S''|/|S| \simeq 0$ we have for all $t \in S - S''$, $u_t(Y(t)) - u_t(X(t)) \simeq \bar{p} \cdot (Y(t) - X(t))$. Let $S_0 = S' \cup S''$, then for all $t \in S - S_0$, $u_t(X(t)) - u_t(Y(t)) \simeq \bar{q} \cdot (X(t) - Y(t)) \simeq \bar{p} \cdot (X(t) - Y(t))$ and $|S_0|/|S| \simeq 0$.

For all $t \in T$ and for all $\bar{x} \in {}^*\Omega_d$, $u_t(\bar{x}) - u_t(X(t)) \lesssim \bar{p} \cdot (\bar{x} - X(t))$. Hence for all $t \in S - S_0$,

$$\begin{aligned} u_t(\bar{x}) - u_t(Y(t)) &= u_t(\bar{x}) - u_t(X(t)) + u_t(X(t)) - u_t(Y(t)) \\ &\lesssim \bar{p} \cdot (\bar{x} - X(t)) + \bar{p} \cdot (X(t) - Y(t)) \\ &\lesssim \bar{p} \cdot (\bar{x} - Y(t)). \end{aligned}$$

Fix j with $1 \leqq j \leqq d$. Since $(1/|S|)\Sigma_{t \in S} Y(t) \gtrsim \bar{0}$ and $|S_0|/|S| \simeq 0$, there exists a $t_1 \in S - S_0$ with $Y_j(t_1) \not\simeq 0$. Thus, by part (ii) of Lemma 1, $\nabla_j u_{t_1}(Y(t_1)) = \bar{q}_j$. On the other hand, ${}^\circ u_{t_1}({}^\circ Y(t_1) + \delta \bar{e}_j) - {}^\circ u_{t_1}({}^\circ Y(t_1)) \leqq {}^\circ \bar{p}_j \delta$ for any real δ with $|\delta| < |{}^\circ Y_j(t_1)|$. Thus, $\bar{p}_j \simeq \nabla_j {}^\circ u_{t_1}({}^\circ Y(t_1)) \simeq \bar{q}_j$ for each $j \leqq d$.

LEMMA 6. *Given a family of utilities $\{u_t\}_{t \in T}$ as in Proposition 1, let V be the corresponding game and let $(X, \bar{p}) \in \mathcal{M}(T, I, \{u_t\}_{t \in T})$. Let S be a subset of T such that $|S| \in {}^*N - N$, X is feasible on S and $(1/|S|)\Sigma_{t \in S} X(t) \gtrsim \bar{0}$. Then for every $t_0 \in S$,*

$$V(S) - V(S - \{t_0\}) = \frac{1}{\omega}[u_{t_0}(X(t_0)) + \bar{p} \cdot (I(t_0) - X(t_0)) + \varepsilon]$$

where $\varepsilon \simeq 0$.

PROOF. Let $(Y, \bar{q}) \in \mathcal{M}(S, I, \{u_t\}_{t \in T})$ and let $(Z, \bar{l}) \in \mathcal{M}(S - \{t_0\}, I, \{u_t\}_{t \in T})$. By Lemma 5, $\bar{q} \simeq \bar{p} \simeq \bar{l}$. Moreover,

$$Y(t_0) + \sum_{t \in S - \{t_0\}} Y(t) = I(t_0) + \sum_{t \in S - \{t_0\}} I(t)$$

and therefore

$$\sum_{t\in S-\{t_0\}}(Y(t)-Z(t))=\sum_{t\in S-\{t_0\}}(Y(t)-I(t))=I(t_0)-Y(t_0).$$

By Lemma 2, there is an allocation W such that for each $t\in S-\{t_0\}$ and $j\leqq d$, $W_j(t)\simeq 0$ and if $Z_j(t)\simeq 0$, $W_j(t)=0$; moreover $\Sigma_{t\in S-\{t_0\}}W(t)=I(t_0)-Y(t_0)$ and

$$\sum_{t\in S-\{t_0\}}u_t(Z(t)+W(t))-u_t(Z(t))\simeq \bar{l}\cdot(I(t_0)-Y(t_0)).$$

Therefore

$$\sum_{t\in S-\{t_0\}}u_t(Y(t))\quad-\sum_{t\in S-\{t_0\}}u_t(Z(t))\gtrsim \bar{l}\cdot(I(t_0)-Y(t_0))\simeq\bar{p}\cdot(I(t_0)-Y(t_0)).$$

Similarly, however,

$$\sum_{t\in S-\{t_0\}}u_t(Z(t))\quad-\sum_{t\in S-\{t_0\}}u_t(Y(t))\gtrsim \bar{q}\cdot(Y(t_0)-I(t_0))\simeq-\bar{p}\cdot(I(t_0)-Y(t_0)).$$

Hence,

$$\sum_{t\in S}u_t(Y(t))-\sum_{t\in S-\{t_0\}}u_t(Z(t))$$
$$=u_{t_0}(Y(t_0))+\sum_{t\in S-\{t_0\}}u_t(Y(t))-u_t(Z(t))\simeq u_{t_0}(Y(t_0))+\bar{p}\cdot(I(t_0)-Y(t_0)).$$

But $u_{t_0}(Y(t_0))\simeq u_{t_0}(X(t_0))+\bar{p}\cdot(Y(t_0)-X(t_0))$. This follows from the fact that $\bar{p}\simeq\bar{q}$ and Proposition 1. Hence

$$\omega(V(S)-V(S-\{t_0\}))=\sum_{t\in S}u_t(Y(t))-\sum_{t\in S-\{t_0\}}u_t(Z(t))$$
$$\simeq u_{t_0}(X(t_0))+\bar{p}\cdot(I(t_0)-X(t_0)).$$

LEMMA 7. *For any $S\subset T$ and any $t_0\in S$, $\omega(V(S)-V(S-\{t_0\}))$ is finite.*

PROOF. Choose $t_1\in S-\{t_0\}$, and let $I'(t)=I(t)$ for $t\neq t_1$, and $I'(t_1)=I(t_1)+I(t_0)$. Let $(Y,\bar{q})\in\mathcal{M}(S-\{t_0\},I',\{u_t\}_{t\in T})$. Let W be an assignment such that for each $t\in S-\{t_0\}$, $\bar{0}\leqq W(t)\leqq Y(t)$ and $\Sigma_{t\in S-\{t_0\}}W(t)=I(t_0)$. By Lemma 1, the uniform boundedness of the gradients of the u_t's on compact sets, and the mean value theorem, there is a real number $m>0$ such that for each $t\in S-\{t_0\}$, $u_t(Y(t)-W(t))-u_t(Y(t))\geqq -m\,\|W(t)\|_2$, where $\|\cdot\|_2$ denotes the Euclidean norm. Let B be a finite upper bound for the u_t's. Then

$$\omega(V(S)-V(S-\{t_0\}))$$

$$\leqq B+\sum_{t\in S-\{t_0\}} u_t(Y(t))-\sum_{t\in S-\{t_0\}} u_t(Y(t)-W(t))$$

$$\leqq B+m\sum_{t\in S-\{t_0\}}\sum_{j=1}^{d} W_j(t)=B+m\sum_{j=1}^{d} I_j(t_0).$$

Note that Lemma 7 implies $\max_{S\subset T, t_0\in S}[\omega(V(S)-V(S-\{t_0\}))]$ is finite.

LEMMA 8. *Let $(X,p)\in\mathcal{M}(T,I,\{u_t\}_{t\in T})$ and let $n\in {}^*N-N$. Then almost all coalitions $S\subset T$ with $|S|=n$ have the property that $(1/n)\Sigma_{t\in S}X(t)\simeq(1/n)\Sigma_{t\in S}I(t)\gtrsim\bar{0}$. (Here "almost all" means that if $\mathcal{S}$ is the set of coalitions of size n and $\mathcal{S}_0$ is the subset for which the theorem is false, then $|\mathcal{S}_0|/|\mathcal{S}|\simeq 0$.)*

PROOF. Let $\delta=n^{-\frac{1}{4}}$. For each $k\in N$, $k^{2d}\delta\simeq 0$, where d is the dimension of the commodity space. Fix $\gamma\in{}^*N-N$ so that $\gamma^{2d}\delta\simeq 0$. Let $L=\max_{t\in T}(\|X(t)\|+\|I(t)\|+1)$, and let B be the box defined by

$$B=\{\bar{x}\in{}^*\Omega_d : 0\leqq\bar{x}_j<L\ \forall j\leqq d\}.$$

Let $\mathcal{B}$ be the set of infinitesimal boxes of the form

$$\left\{\bar{x}\in B:\frac{k_j}{\gamma}L\leqq\bar{x}_j<\frac{k_j+1}{\gamma}L\right\},$$

where for each $j\leqq d$, $0\leqq k_j\leqq\gamma-1$, and $k_j\in{}^*N$.

Divide T into equivalence classes called types: t_1 and t_2 are of the same type if $X(t_1)$ and $X(t_2)$ are in the same box in $\mathcal{B}$ and if $I(t_1)$ and $I(t_2)$ are in the same box in $\mathcal{B}$. Let m be the number of types in T; clearly $m\leqq\gamma^{2d}$. Order the types and let r_i, $1\leqq i\leqq m$, be the ratio of the number of elements of the i-th type to $|T|=\omega$.

By Chebyschev's inequality, the probability is at most $1/4n\delta^2$ of obtaining an internal random sample S of size n from T for which the ratio s_i of the number of elements of type i in S to n differs from r_i by at least δ. That is,

$$P(|s_i-r_i|\geqq\delta)\leqq 1/4n\delta^2=\delta^2/4,$$

since $\delta\equiv n^{-\frac{1}{4}}$. Let $\mathcal{S}=\{S\subset T:|S|=n\}$. We have shown that there is a set $\mathcal{S}_0\subset\mathcal{S}$ with $|\mathcal{S}_0|/|\mathcal{S}|\leqq(\delta^2/4)\gamma^{2d}\simeq 0$ so that for each $S\in\mathcal{S}-\mathcal{S}_0$, $\max_i|s_i-r_i|<\delta$.

Choose one representative $t_i\in T$ for each type. Using the ordering on T, let t_{ij} be the j-th element in the i-th type; let $\bar{\alpha}_{ij}=X(t_{ij})-X(t_i)$ and $\bar{\beta}_{ij}=I(t_{ij})-I(t_i)$. Clearly, $\|\bar{\alpha}_{ij}\|\simeq 0$ and $\|\bar{\beta}_{ij}\|\simeq 0$. Given $S\in\mathcal{S}$, and s_i, the ratio of the number of elements in S of type i to n, let j_i run through the elements of type i in S when $s_i \neq 0$. Then

$$\begin{aligned}
&\frac{1}{n}\sum_{t\in S} X(t) - I(t) \\
&= \frac{1}{n}\sum_{i:s_i\neq 0}\sum_{j_i=1}^{ns_i}[(X(t_i)+\bar{\alpha}_{ij_i}) - (I(t_i)+\bar{\beta}_{ij_i})] \\
&= \sum_{i=1}^{m} s_i(X(t_i) - I(t_i)) + \frac{1}{n}\sum_{i:s_i\neq 0}\sum_{j_i=1}^{ns_i}(\bar{\alpha}_{ij_i} - \bar{\beta}_{ij_i}).
\end{aligned}$$

Since n is at least the number of terms in the second sum on the right side of the last equation and the average of infinitesimal vectors is infinitesimal, we have

$$\frac{1}{n}\sum_{t\in S}(X(t) - I(t)) \simeq \sum_{i=1}^{m} s_i(X(t_i) - I(t_i)).$$

By similar reasoning, we have

$$\bar{0} = \frac{1}{\omega}\sum_{t\in T} X(t) - I(t) \simeq \sum_{i=1}^{m} r_i(X(t_i) - I(t_i)).$$

Now if $S \in \mathscr{S} - \mathscr{S}_0$, then

$$\begin{aligned}
\left\|\frac{1}{n}\sum_{t\in S}(X(t) - I(t))\right\| &\simeq \left\|\sum_{i=1}^{m}(r_i + (s_i - r_i))(X(t_i) - I(t_i))\right\| \\
&\lesssim \max_i |s_i - r_i| \cdot \max_i \|X(t_i) - I(t_i)\| \cdot m \\
&\lesssim \delta\gamma^{2d} \cdot \max_i \|X(t_i) - I(t_i)\| \simeq 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{1}{\omega}\sum_{t\in T} I(t) &= \frac{1}{\omega}\sum_{i=1}^{m}\sum_{j_i=1}^{\omega r_i}(I(t_i) + \bar{\beta}_{ij_i}) \\
&\simeq \sum_{i=1}^{m} r_i I(t_i) \gg \bar{0},
\end{aligned}$$

and similarly,

$$\frac{1}{n}\sum_{t\in S} I(t) \simeq \sum_{i=1}^{m} s_i I(t_i).$$

Since

$$\left\| \sum_{i=1}^{m} (s_i - r_i) I(t_i) \right\| \leqq \delta \gamma^{2d} \cdot \max_i \| I(t_i) \| \approx 0,$$

$$\frac{1}{n} \sum_{t \in S} I(t) \gg \bar{0}.$$

PROPOSITION 2. *Given a family of utilities* $\{u_t\}_{t \in T}$ *as in Proposition* 1 *and the corresponding game* V, *if* $(X, \bar{p}) \in \mathcal{M}(T, I, \{u_t\}_{t \in T})$ *then for all* $t_0 \in T$ *and for "almost all"* $S \subset T$ *we have* $\omega(V(S) - V(S - \{t_0\})) \approx u_{t_0}(X(t_0)) + \bar{p} \cdot (I(t_0) - X(t_0))$. *Thus,*

$$\varphi_V(t_0) = E[V(S) - V(S - \{t_0\})] = \frac{1}{\omega} [u_{t_0}(X(t_0)) + \bar{p} \cdot (I(t_0) - X(t_0)) + \varepsilon],$$

where $\varepsilon \approx 0$. *(Here* E *is the expectation operator for the probability measure on the class of the subsets of* T *which contain* t_0 *such that each size is equally likely and for any given size, each set is equally likely.)*

PROOF. The formula for the Shapley-value is given by transfer as

$$\varphi_V(t_0) = \frac{1}{\omega} \sum_{j=1}^{\omega-1} \frac{1}{\binom{\omega - 1}{j}} \left(\sum_{\substack{R \subset T - \{t_0\} \\ |R| = j}} [V(R \cup \{t_0\}) - V(R)] \right).$$

Fix $k \in {}^*N - N$ such that $k/\omega \approx 0$. Since by Lemma 7, $\omega(V(R \cup \{t_0\}) - V(R))$ has a finite upper bound, we have

$$\omega \varphi_V(t_0) \approx \frac{1}{\omega} \sum_{j=k}^{\omega-1} \frac{\omega}{\binom{\omega - 1}{j}} \left(\sum_{\substack{R \subset T - \{t_0\} \\ |R| = j}} [V(R \cup \{t_0\}) - V(R)] \right).$$

For infinite coalitions R, e.g., $|R| \geqq k$, we can apply Lemmas 6, 7, and 8 and obtain for fixed $j \geqq k$

$$\frac{\omega}{\binom{\omega - 1}{j}} \sum_{\substack{R \subset T - \{t_0\} \\ |R| = j}} [V(R \cup \{t_0\}) - V(R)] \approx u_{t_0}(X(t_0)) + \bar{p} \cdot (I(t_0) - X(t_0)).$$

Therefore

$$\omega \varphi_V(t_0) \approx \frac{\omega - k}{\omega} [u_{t_0}(X(t_0)) + \bar{p} \cdot (I(t_0) - X(t_0))]$$

$$\approx u_{t_0}(X(t_0)) + \bar{p} \cdot (I(t_0) - X(t_0)).$$

We now prove part b of the Theorem: If Y is a value allocation with respect to an appropriate family of utilities, then Y is a competitive allocation.

If Y is a value allocation with respect to a family of utilities as in Proposition 1, then by definition

$$\frac{1}{\omega}\sum_{t\in T} u_t(Y(t)) \simeq \sum_{t\in T} \varphi_V(t) = V(T) = \frac{1}{\omega}\sum_{t\in T} u_t(X(t))$$

where $(X, \bar{p}) \in \mathcal{M}(T, I, \{u_t\}_{t\in T})$. Thus there exists by Lemma 3 a set $T_0 \subset T$ with $|T_0|/|T| \simeq 0$ such that for all $t \in T - T_0$, $u_t(Y(t)) \simeq u_t(X(t)) + \bar{p}\cdot(Y(t) - X(t))$. Hence by Proposition 2,

$$u_t(Y(t)) \simeq u_t(X(t)) + \bar{p}\cdot(I(t) - X(t)) + \bar{p}\cdot(Y(t) - I(t))$$

$$\simeq \omega[\varphi_V(t)] + \bar{p}\cdot(Y(t) - I(t)).$$

Therefore,

$$\omega\varphi_V(t) \simeq u_t(Y(t)) + \bar{p}\cdot(I(t) - Y(t))$$

for all $t \in T - T_0$.

We claim that $\bar{p}\cdot Y(t) \simeq \bar{p}\cdot I(t)$ for almost all $t \in T$. Let

$$S_n = \{t \in T - T_0 : \bar{p}\cdot Y(t) \geqq \bar{p}\cdot I(t) + 1/n\}$$

and

$$U_n = \{t \in T - T_0 : \bar{p}\cdot Y(t) \leqq \bar{p}\cdot I(t) - 1/n\}.$$

Then $S_n \subset S_{n+1}$ and $U_n \subset U_{n+1}$ for each $n \in {}^*N$. Assume that $|S_n|/|T - T_0| \geqq 1/n$; then by definition,

$$\frac{1}{\omega}\sum_{t\in S_n} u_t(Y(t)) \simeq \sum_{t\in S_n} \varphi_V(t) = \frac{1}{\omega}\sum_{t\in S_n} (u_t(Y(t)) + \bar{p}\cdot(I(t) - Y(t)) + \varepsilon_t),$$

where $\varepsilon_t \simeq 0$. Thus, $(1/\omega)\Sigma_{t\in S_n}\bar{p}\cdot(I(t) - Y(t)) \simeq 0$. But $\bar{p}\cdot I(t) - \bar{p}\cdot Y(t) \leqq -1/n$ for all $t \in S_n$, so

$$\frac{1}{\omega}\sum_{t\in S_n} \bar{p}\cdot(I(t) - Y(t)) \leqq \frac{|S_n|}{\omega}(-1/n) \lesssim -1/n^2,$$

which is impossible if $n \in N$. Therefore, for some $\nu \in {}^*N - N$, $|S_\nu|/|T - T_0| < 1/\nu$ and similarly $|U_\nu|/|T - T_0| < 1/\nu$. That is, $\bar{p}\cdot Y(t) \simeq \bar{p}\cdot I(t)$ except for $t \in S_\nu \cup U_\nu \cup T_0$. Thus, given $t \in T - [S_\nu \cup U_\nu \cup T_0]$ and $\bar{y} \in {}^*\Omega_d$ with $\bar{p}\cdot\bar{y} \lesssim \bar{p}\cdot I(t)$, we have $\bar{p}\cdot\bar{y} \lesssim \bar{p}\cdot Y(t)$, whence

$$u_t(\bar{y}) - u_t(Y(t)) = [u_t(\bar{y}) - u_t(X(t))] - [u_t(Y(t)) - u_t(X(t))]$$

$$\lesssim \bar{p}\cdot(\bar{y} - X(t)) - \bar{p}\cdot(Y(t) - X(t)) = \bar{p}\cdot(\bar{y} - Y(t)) \lesssim 0.$$

That is, $u_t(\bar{y}) \lesssim u_t(Y(t))$.

We now prove part a of the theorem.

Let $(Y, \bar{p})$ be a competitive equilibrium. Let $\{\tilde{u}_t\}_{t\in T}$ be a representing family of utilities essentially concave in $^*B_\gamma(\bar{0})$ where if $\gamma_1 = [\max_{t\in T}\bar{p}\cdot I(t)/\min_{1\leq j\leq d}\bar{p}_j]$, $\gamma = {}^0(1+\gamma_1)$. Let $X(t)$ be a point maximizing $\tilde{u}_t(\bar{y})$ on the budget hyperplane

$$B_{\bar{p}}(t) = \{\bar{y} \in {}^*\Omega_d : \bar{p}\cdot\bar{y} = \bar{p}\cdot I(t)\} \subset {}^*B_\gamma(\bar{0}).$$

Since $(Y, \bar{p})$ is a competitive equilibrium, there exists $T_0 \subset T$ such that for all $t \in T - T_0$, $\tilde{u}_t(Y(t)) \simeq \tilde{u}_t(X(t))$ and $|T_0|/|T| \simeq 0$.

Given $t \in T$, let j_0 be the first $j \leqq d$ such that $X_{j_0}(t) > 0$, and let $\alpha_t = \bar{p}_{j_0}/\nabla_{j_0}\tilde{u}_t(X(t))$. Then $\alpha_t \not\approx 0$ is finite since $\bar{p} \gg \bar{0}$ and $\nabla_{j_0}\tilde{u}_t(\bar{x}) \not\approx 0$ and $\nabla_{j_0}\tilde{u}_t(\bar{x})$ is finite when $\bar{x} \in B_{\bar{p}}(t)$. Let $u_t = \alpha_t\tilde{u}_t$; then $\nabla_{j_0}u_t(X(t)) = \bar{p}_{j_0}$. Let $\bar{x} = X(t)$, and for $j \neq j_0$, fix $\bar{y} \in {}^*\Omega_d$ with $\bar{p}\cdot\bar{y} = \bar{p}\cdot I$ and $\bar{y}_i = \bar{x}_i$ if $i \neq j$ and $i \neq j_0$. Then

$$\nabla u_t(\bar{x})\cdot(\bar{y}-\bar{x}) = \bar{p}_{j_0}(\bar{y}_{j_0} - \bar{x}_{j_0}) + \nabla_j u_t(\bar{x})\cdot(\bar{y}_j - \bar{x}_j) \leqq 0,$$

since the directional derivative at $X(t)$ along the budget plane is $\leqq 0$. But $\bar{p}_{j_0}(\bar{y}_{j_0} - \bar{x}_{j_0}) = -\bar{p}_j(\bar{y}_j - \bar{x}_j)$, so $\nabla_j u_t(\bar{x})\cdot(\bar{y}_j - \bar{x}_j) - \bar{p}_j(\bar{y}_j - \bar{x}_j) \leqq 0$. If $X_j(t) > 0$, then $\bar{p}_j = \nabla_j u_t(\bar{x})$. Otherwise we have $\nabla_j u_t(\bar{x}) \leqq \bar{p}_j$.

We now have for any $\bar{y} \in {}^*\Omega_d$,

$$u_t(\bar{y}) - u_t(X(t)) \leqq \bar{p}\cdot(\bar{y} - X(t)) = \bar{p}\cdot\bar{y} - \bar{p}\cdot I(t),$$

for all $t \in T$.

Let $(Z, \bar{q}) \in \mathcal{M}(T, I, \{u_t\}_{t\in T})$. Then

$$\sum_{t\in T} u_t(Z(t)) - u_t(X(t)) \leqq \sum_{t\in T}\bar{p}\cdot(Z(t) - I(t)) = \bar{p}\cdot\sum_{t\in T} Z(t) - I(t) = 0,$$

and so

$$\frac{1}{\omega}\sum_{t\in T} u_t(Z(t)) \leqq \frac{1}{\omega}\sum_{t\in T} u_t(X(t)) \simeq \frac{1}{\omega}\sum_{t\in T} u_t(Y(t)).$$

On the other hand,

$$\frac{1}{\omega}\sum_{t\in T}(u_t(Y(t)) - u_t(Z(t))) \lesssim \frac{1}{\omega}\sum_{t\in T}\bar{q}\cdot(Y(t) - Z(t))$$

$$= \bar{q}\cdot\left[\frac{1}{\omega}\sum_{t\in T}(Y(t) - Z(t))\right] = \bar{q}\cdot\left[\frac{1}{\omega}\sum_{t\in T}(Y(t) - I(t))\right] \simeq 0.$$

Hence $(1/\omega)\sum_{t\in T}u_t(Z(t)) \simeq (1/\omega)\Sigma_{t\in T}u_t(Y(t))$ which implies that $(1/\omega)\Sigma_{t\in T}u_t(Z(t)) \simeq (1/\omega)\Sigma_{t\in T}u_t(X(t))$. Since

$$\bar{p}\cdot\left(\frac{1}{|T|}\sum_{t\in T}(X(t) - Z(t))\right) = \bar{p}\cdot\left(\frac{1}{|T|}\sum_{t\in T}(X(t) - I(t))\right) = 0,$$

we can use the proof of Lemma 3 to show that for almost all $t \in T$,

$$u_t(Z(t)) - u_t(X(t)) \simeq \bar{p} \cdot (Z(t) - X(t)).$$

But for all $t \in T$ and all $\bar{x} \in {}^*\Omega_d$,

$$u_t(\bar{x}) - u_t(X(t)) \leqq \bar{p} \cdot (\bar{x} - X(t));$$

hence $u_t(\bar{x}) - u_t(Z(t)) \lesssim \bar{p} \cdot (\bar{x} - Z(t))$ for almost all $t \in T$. Using the argument at the end of the proof of Lemma 5, we see that $\bar{p} \simeq \bar{q}$.

By Proposition 2, if V is the game associated with $\{u_t\}_{t \in T}$ then for all $t \in T$,

$$\omega[\varphi_V(t)] \simeq u_t(Z(t)) + \bar{q} \cdot (I(t) - Z(t)).$$

But for almost all $t \in T$,

$$u_t(Z(t)) + \bar{q} \cdot (I(t) - Z(t)) \simeq u_t(Z(t)) + \bar{p} \cdot (X(t) - Z(t)) + \bar{p} \cdot (I(t) - X(t)) \simeq u_t(X(t)).$$

Therefore, $\omega[\varphi_V(t)] = u_t(Y(t)) + \varepsilon_t$ where $\varepsilon_t \simeq 0$ for almost all $t \in T$ and ε_t is finite for all $t \in T$ by Proposition 2.

Hence for all $S \subset T$, $\Sigma_{t \in S} \varphi_V(t) \simeq (1/\omega) \Sigma_{t \in S} u_t(Y(t))$. That is, $Y(t)$ is a value allocation.

This completes the proof.

References

1. R. J. Aumann, *Values of markets with a continuum of traders*, Econometrica **43** (1975), 611–646.
2. R. J. Aumann and L. Shapley, *Values of Non-Atomic Games*, Princeton University Press, Princeton, 1974.
3. D. J. Brown and A. Robinson. *Nonstandard exchange economies*, Econometrica **43** (1974), 41–55.
4. D. J. Brown and A. Robinson, *The cores of large standard exchange economies*, J. Economic Theory **9** (1974), 245–254.
5. D. J. Brown, *Existence of a competitive equilibrium in a nonstandard exchange economy*, Econometrica **44** (1976), 537–546.
6. G. Debreu, *Smooth preferences*, Econometrica **40** (1972), 603–616.

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